NASA TECHNICAL MEMORANDUM

64

NASA TM

Report No. 53909

PRODUCT MEAN VALUES AND CONVECTION SPEED

By W. H. Heybey
Aero-Astrodynamics Laboratory

September 12, 1969

NASA

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama

MSFC - Form 3190 (September 1968)

NASA - GEORGE C. MARSHALL SPACE FLIGHT CENTER
changed to TM X-53909 on September 12, 1969

IN-AERO-67-8

November 20, 1967

PRODUCT MEAN VALUES AND CONVECTION SPEED

By

W. H. Heybey

TECHNICAL AND SCIENTIFIC STAFF
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

IN-AERO-67-8

PRODUCT MEAN VALUES AND CONVECTION SPEED

By

W. H. Heybey

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

In crossed-beam methodology the relationship of the beam intensity covariances and local extinction coefficient covariances comes up as an important problem. The approach leading to it has been carefully restated in a systematic review. The answer suggested places more weight on experimental determination of correlation volumes and flow direction than on mathematical conditions imposed a priori on the turbulent state. Application is made to the calculation of convection velocity.

IN-AERO-67-8

PRODUCT MEAN VALUES AND CONVECTION SPEED

SUMMARY

The physical conditions under which basic mathematics currently in use are valid have been carefully specified. If they are realized, crossed-beam examination of a thin turbulent layer promises to yield reliable results. In thick layers the turbulent correlation must remain restricted to small volumes if one wants to calculate two-point values from two-beam values. The experimentation normally necessary to explore the flow's correlation structure can be forgone in computing bulk velocity, if one agrees to define it by the time of the maximum two-beam value and the beam distance in flow direction. The latter must be ascertained as (approximately) constant simultaneously with any bulk speed measurement.

INTRODUCTION

The relationship of the two-beam and two-point product mean values is considered a basic issue in crossed-beam experimentation. Since it has been controversial, and still is, to some extent, an attempt is made to examine the case in a systematic way, starting out with the very foundations of the crossed-beam method in order to ascertain the physical significance of the mathematical formulations that follow from them. Determination of the convection speed is always kept in view, as it is a main application.

ORIENTATION

Two non-parallel light beams, a and b, of infinitesimal width are sent through turbulent flow at some distance from each other. Let us mark on them the points A and B as the origins of running coordinates, α and β, defining the points on the a- and b-beams. The flow state along the beams will have to be described by functions that depend on either α or β and on time.

It will be assumed from the outset that the convection velocity has a constant direction given by the unit vector $\underline{v}$. If it is not parallel to the pair of parallel planes that is established by the lines a and b (as we will postulate), there exists exactly one pair of points, P_A and and P_B, on the beams such that $\overrightarrow{P_A P_B} \parallel \underline{v}$. Since these points are in line with the flow, the physical correlation near them will be stronger than near any other pair. We select them as the origins A and B, both of which then have a definite location. It follows that their connecting line will not in general be normal to a or to b.

We wish to inquire into what can be learned through the modifications of the beam intensities alone, and are not therefore concerned with the state of the flow outside the beams. To view the a- and b-lines as parallel to two axes of some rectilinear three-dimensional system is not necessary as a consequence, nor has it been found advisable to do so. Especially, the line AB = s is not considered as parallel to or coinciding with a third axis. While the distance s is constant for every experimental run, α and β are true variables, as the light is acted upon at every point P_A, and at every point P_B.

A beam, operating for a time span of T_o seconds, can be imagined as an infinitely dense sequence of separate instantaneous "flashes." A flash "at the time t," needing practically no time to travel from the source to a detector, will nevertheless be weakened through various absorption and scattering mechanisms that it encounters on its way, e.g., along the a-trace. Its intensity, I_a, recorded "at time t," varies with t, since in a turbulent flow those mechanisms work in a slightly different manner on succeeding flashes. The detector record will exhibit the function $I_a(t)$ as a random curve over time. The temporal mean, taken at time t*, will, with sufficiently persistent turbulent characteristics, become practically independent of t* after t* has grown sufficiently large, say, from t* = T* on. Mathematically, this type of turbulence is described by what may be termed the "persistency" condition:

$$\bar{I}_a = \frac{1}{T} \int_o^T I_a(t)\, dt \approx \text{const.} \quad \text{for } T \geqq T^*. \tag{1}$$

Failure of a record to comply with this condition may be caused either by too short an operating time* (which deficiency might be amended) or by macroscopic time trends in the turbulent state for which there is no remedy.

*It is clear that T cannot be taken larger than T_o.

It may be mentioned in passing that, if a whole series of a-records of lengths T_0 were taken in rapid succession, there is no guaranty that the ensemble means should be the same at all times (i.e., stationary), merely because, in every sample, condition (1) is satisfied.

The absorptive weakening of light intensity is described in optics by the exponential law

$$dI = -K\ I\ d\alpha. \tag{2}$$

In a homogeneous gas (as considered in optics) the absorption coefficient K is a constant for a given wavelength and refractive index (density). The latter varies in steady (but not uniform) motion, so that the gas is "homogeneous" in its smallest parts only; in turbulent motion, it is so only instantaneously. While maintaining the law (2) and settling for a definite wavelength*, we must here regard K as a function both of α and the time t at which the flash occurs. The latter's intensity, I, along the a-beam likewise is not only a function of flash time, but also of location, $I = I(\alpha, t)$. If α_1 and α_2 are the coordinates of the sender and detector, respectively, $I(\alpha_1, t) = I_s(t)$, $I(\alpha_2, t) = I_a(t)$. The first formula allows for possible time fluctuations of the radiated intensity. However, for the purpose of the present theoretical investigation, ideal operation will be presupposed: there are no power fluctuations at the source, no instrumentation or other noise effects. Thus, $I_s(t) = I_s =$ constant.

The law (2) will be assumed to hold as well for the local intensity loss caused by any other dissipation process, as by molecular** or particle scattering; K will therefore be given the more general name of "extinction coefficient."

---

* One can use white light as well. This is not done in optics, because there the interest precisely centers about the questions which frequencies are absorbed and to what degree.

** Lord Rayleigh has found that here

$$K \sim \frac{(n^2 - 1)^2}{N\lambda^4}$$

(n = refractive index, N = number density, λ = wavelength).

On integrating the differential (2)

$$\frac{I_a(t)}{I_s} = \exp\left\{-\int_{\alpha_1}^{\alpha} K(\alpha, t)\, d\alpha\right\}. \tag{3}$$

The average of I_a over the time span of the integral (1) follows as

$$\bar{I}_a = I_s \exp\left\{-\int_{\alpha_1}^{\alpha_2}\left[\frac{1}{T}\int_0^T K(\alpha, t)\, dt\right] d\alpha \equiv I_s \exp\left\{-\int_{\alpha_1}^{\alpha_2} \bar{K}(\alpha)\, d\alpha\right\}, \tag{4}$$

where $\bar{K}(\alpha)$ is the time average of K at station α. T could be taken here as equal to the operating time T_o; but it is not advisable to do so. With the b-beam, a time delay τ will be contemplated, while the integration still has to extend from $t = 0$ to $t = T$.

If the difference

$$k(\alpha, t) = K(\alpha, t) - \bar{K}(\alpha) \tag{5}$$

is introduced, the ratio (3) becomes

$$\frac{I_a(t)}{I_s} = \exp\left\{-\int_{\alpha_1}^{\alpha_2} k(\alpha, t)\, d\alpha\right\} \cdot \exp\left\{-\int_{\alpha_1}^{\alpha_2} \bar{K}(\alpha)\, d\alpha\right\},$$

and, with the use of relation (4), shifts into the ratio

$$\frac{I_a(t)}{\bar{I}_a} = \exp\left\{-\int_{\alpha_1}^{\alpha_2} k(\alpha, t)\, d\alpha\right\} = 1 - \int_{\alpha_1}^{\alpha_2} k\, d\alpha + \frac{1}{2!}\left[\int_{\alpha_1}^{\alpha_2} k\, d\alpha\right]^2 \mp \ldots\,. \tag{6}$$

Suppose now that the turbulent motion is such that the assumption

$$\left| \int_{\alpha_1}^{\alpha_2} k(\alpha, t) \; d\alpha \right| \ll 1 \qquad (7)$$

can be made for every time t. Then it follows from the ratio (6) that

$$i(t) \equiv \bar{I}_a - I_a(t) = \bar{I}_a \int_{\alpha_1}^{\alpha_2} k(\alpha, t) \; d\alpha. \qquad (8)$$

It is seen that the linearizing condition (7) implies that the recorded intensity $I_a(t)$ should never stray far from its mean value. This does not necessarily exclude rather large values of $k(\alpha, t)$, that is, marked deviations of the local value $K(\alpha, t)$ from its temporal mean $\bar{K}(\alpha)$, since for every flash they might effectively cancel out when integrated over α. But, whether the a-beam cuts through vigorously fluctuating or relatively calm flow, if large-scale temporal trends in the extinction processes are present at many or all points P_A, the detector record will reveal that $I_a(t)$ is not close to $\bar{I}_a$ everywhere. Linearization is then dubious, if not prohibited. The persistency condition (1) which is likely to be violated in such records with long-term up-and-down or monotonic trends, can be expected, on the other hand, to be satisfied if but small deviations occur from the intensity mean. We may say that if the persistency condition is observed as realized, the linearizing assumption (7) may be taken for granted. Even if the record fails to comply with this condition, linearization still may involve no risks when long-term temporal trends are not apparent. (Those trends are sometimes classified as non-stationary -- another use of an overburdened word.)

As an immediate consequence of definition (5),

$$\int_{0}^{T} k(\alpha, t) \; dt = 0. \qquad (9)$$

Both the requirements (7) and (9) should be heeded when setting up approximating expressions for the function $k(\alpha, t)$. It is already recognized that T should as a rule be smaller than T_o. The limits of the integral (7) can be narrowed down in circumstances. It is not necessary that the turbulent realm should extend all the way from sender to detector, provided that outside it the extinction coefficient is constant with respect to time. Let it be equal to $K_1(\alpha)$ in $\alpha_1 \leq \alpha \leq \alpha_1'$, to $K_2(\alpha)$ in $\alpha_2' \leq \alpha \leq \alpha_2$. Then

$$\frac{I_a(t)}{I_s} = \exp\left\{-\int_{\alpha_1}^{\alpha_1'} K_1(\alpha)\, d\alpha - \int_{\alpha_1'}^{\alpha_2'} K(\alpha, t)\, d\alpha - \int_{\alpha_2'}^{\alpha_2} K_2(\alpha)\, d\alpha\right\}$$

and

$$\frac{\bar{I}_a}{I_s} = \exp\left\{-\int_{\alpha_1}^{\alpha_1'} K_1(\alpha)\, d\alpha - \int_{\alpha_1'}^{\alpha_2'} \bar{K}(\alpha)\, dt - \int_{\alpha_2'}^{\alpha_2} K_2(\alpha)\, d\alpha\right\} ,$$

so that

$$\frac{I_a(t)}{\bar{I}_a} = \frac{\exp\left\{-\int_{\alpha_1'}^{\alpha_2'} K(\alpha, t)\, d\alpha\right\}}{\exp\left\{-\int_{\alpha_1'}^{\alpha_2'} \bar{K}(\alpha)\, d\alpha\right\}} = 1 - \int_{\alpha_1'}^{\alpha_2'} k(\alpha, t)\, d\alpha,$$

in accordance with expression (6). Definition (5) and condition (7) now refer to the turbulent interval $< \alpha_1', \alpha_2' >$ alone. In the future, the integration limits α_1, α_2 will be regarded as the bounds of the turbulent segment on the a-beam. The values K_1 and K_2 in the laminar parts are of no consequence, provided that they truly depend on α alone: The outside flow must be steady; number and size of scattering particles are not allowed to vary with time along the laminar segments.

Analogous expressions evolve with the beam b under the conditions stated. In writing them down, primes will be used if necessary for clarity. Thus

$$\bar{I}_b(\tau) = \frac{1}{T} \int_0^T I_b(t + \tau)\, dt,$$

but

$$i'(t + \tau) = \bar{I}_b(\tau) - I_b(t + \tau) = \bar{I}_b(\tau) \int_{\beta_1}^{\beta_2} k'(\beta,\ t + \tau)\, d\beta, \quad \text{etc.}$$

The crossing of the beams is performed, not indeed physically in general, but mathematically by forming the temporal covariance of $I_a(t)$ and $I_b(t + \tau)$: the value of i at the station t of the a-record is coupled with the value of i' at the station $t + \tau$ of the downstream b-record. The delay time τ is needed for later applications and has the character of a parameter on which the covariance (or "two-beam product mean value") depends:

$$R(\tau) = \frac{1}{T} \int_0^T i(t)\ i'(t + \tau)\, dt =$$

$$= \frac{\bar{I}_a\ \bar{I}_b(\tau)}{T} \int_0^T \left[\int_{\alpha_1}^{\alpha_2} k(\alpha, t)\, d\alpha \cdot \int_{\beta_1}^{\beta_2} k'(\beta, t + \tau)\, d\beta \right] dt \equiv$$

$$\equiv \bar{I}_a\ \bar{I}_b(\tau)\ \tilde{R}(\tau). \tag{10}$$

When the two records are laid above each other with coinciding time scales and time scale zeros, it is understood in this formulation that the b-record is used from $t = \tau$ to $t = T + \tau$, with a number of parametric τ-values. These of course have an upper bound $\tau_u = T_o - T$, while the lower bound is $\tau = 0$.*

The function $\widetilde{R}(\tau)$ introduced above may be written as

$$\widetilde{R}(\tau) = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \left[\frac{1}{T} \int_{0}^{T} k(\alpha, t)\ k'(\beta,\ t + \tau)\ dt \right] d\beta d\alpha\ . \tag{11}$$

The rearrangement of the integration sequence is permitted, since both the integrands and the integration limits are finite. One recalls that condition (7) and the analogous condition for $k'(\beta,\ t + \tau)$ require that the integrand (10) is numerically very small.

The functions $k(\alpha, t)$ and $k'(\beta,\ t + \tau)$ describe, relative to a temporal mean, the time fluctuations of the extinction coefficients at the two points $P_A(\alpha)$ and $P_B(\beta)$. Accordingly, one may define temporal two-point product mean values

$$R_k(\alpha, \beta; \tau) = \frac{1}{T} \int_{0}^{T} k(\alpha, t)\ k'(\beta,\ t + \tau)\ dt, \tag{12}$$

so that the function $\widetilde{R}(\tau)$ may be set into the form

$$\widetilde{R}(\tau) = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} R_k(\alpha, \beta; \tau)\ d\beta d\alpha. \tag{13}$$

R_k is the covariance of $K(t)$ and $K'(t + \tau)$ calculated at the points $P_A(\alpha)$ and $P_B(\beta)$. If the choice of these points is inconsequential,

---

*Negative values of τ must be used when the a-record rather than the b-record is being "delayed."

i.e., if R_k does not actually depend on α and β, so that

$$R_k(\alpha, \beta; \tau) = R_k(0,0;\tau) = \text{const.},$$

we have the simple and desirable relationship

$$\tilde{R}(\tau) = (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\, R_k(0,0;\tau), \tag{14}$$

which shows the two-point mean values that would be found at A and B (or at any other two points P_A and P_B) as proportional to the observable values $\tilde{R}(\tau)$. That turbulence should ever be so organized in toto appears as highly improbable. It is, however, one prime aim of the crossed-beam method to gain local information in spite of the use of integrating light flashes. The question then arises: In what circumstances is it possible to relate the two-beam values $\tilde{R}(\tau)$ to the unknown two-point values $R_k(0,0;\tau)$ by means of τ-independent factors of proportionality?

SHEET TURBULENCE

By definition, both the points A and B (and their connecting line as well) are inside the turbulence region. Let us assume that by chance they occupy the centers of the respective turbulent stretches, whose boundaries then may be given as

$$\alpha_1 = -\alpha^*, \quad \alpha_2 = \alpha^*; \quad \beta_1 = -\beta^*, \quad \beta_2 = \beta^*. \tag{15}$$

This simplifies the results without blotting out essential features. The distribution at flash time t of the k-values can be approximated by the trigonometric polynomial

$$k(\alpha, t) = \sum_{m=0}^{M} [a_m(t) \cos m\alpha + \alpha_m(t) \sin m\alpha]. \tag{16}$$

If M is a sufficiently large number, one may presume that the polynomial will yield a satisfactory approximation to the distribution at all times t. The values of the fluctuations $a_m(t)$ and $\alpha_m(t)$ theoretically can be determined for a given turbulence field by running the gamut of all t's and finding the best approximation for every single flash along the a-trace.

We postulate that $k(\alpha, t)$ satisfies the conditions (7) and (9).

In the same vein we define

$$k'(\beta, t + \tau) = \sum_{n=0}^{N} [b_n(t + \tau) \cos n\beta + \beta_n(t + \tau) \sin n\beta]. \quad (17)$$

From these definitions we obtain the expression

$$R_k(0,0;\tau) = \frac{1}{T} \int_o^T \sum_m a_m(t) \cdot \sum_n b_n(t + \tau)\, dt. \quad (18)$$

On the other hand, with the limits (15) the function $\tilde{R}(\tau)$ becomes

$$\tilde{R}(\tau) = \frac{4}{T} \int_o^T \sum_m a_m(t) \frac{\sin m\alpha^*}{m} \cdot \sum_n b_n(t + \tau) \frac{\sin n\beta^*}{n}\, dt. \quad (19)$$

There is obviously no constant factor of proportionality that would connect $R_k(0,0;\tau)$ to $\tilde{R}(\tau)$ for all values of τ except when both α^* and β^* are very small. Then

$$\tilde{R}(\tau) = 4\alpha^*\beta^* R_k(0,0;\tau). \quad (20)$$

The factor $4\alpha^*\beta^*$ is the product of the turbulent segments along the a- and b-beams. One can show that, if the origins A and B are not their center points, the result (20) goes into the relation (14) when again the segments are very short. It does not matter then, where A and B are located, $R_k(0,0;\tau)$ is always equal to $\tilde{R}(\tau)$ divided by the product of the turbulent segments. (With approximations for k and k' other than by trigonometric polynomials the divisor may turn out merely proportional to this product.)

For defining dimensionless (normalized) expressions let us introduce a third beam, c, into the experimentation which physically crosses the a-beam at the point A (doubling as origin of a linear γ-coordinate), and which often (though not necessarily so) will be chosen as parallel to the beam b. If the a- and c-records are correlated without time delay we may introduce a fluctuation

$$k''(\gamma,t) = \sum_{p=0}^{P} [c_p(t) \cos p\gamma + \gamma_p(t) \sin p\gamma].$$

Note that even with $c \parallel b$ and $P = N$ the coefficients $c_i(t)$, $\gamma_i(t)$ cannot in general be supposed to be equal to $b_i(t + 0)$, $\beta_i(t + 0)$. The distance s being zero with the a- and c-beams, we will use a corresponding subscript in writing down the analogues to the formulas (18), (19), and (20):

$$R_{k,o}(\alpha = 0,\ \gamma = 0;\ \tau = 0) = \frac{1}{T} \int_o^T \sum_m a_m(t) \cdot \sum_p c_p(t)\ dt$$

$$\tilde{R}_o(\tau = 0) = \frac{4}{T} \int_o^T \sum_m a_m(t) \frac{\sin m\alpha^*}{m} \cdot \sum_p c_p(t) \frac{\sin p\gamma^*}{p}\ dt\ .$$

In the event that α^* and γ^* are very small

$$\tilde{R}_o(0) = 4\alpha^*\gamma^*\ R_{k,o}(0,0;0).$$

The normalized values of $R_k(0,0;\tau)$ and $R(\tau)$ will be defined as

$$Q_k = \frac{R_k(0,0;\tau)}{R_{k,o}(0,0;0)} \quad \text{and} \tag{21}$$

$$Q = \frac{R(\tau)}{R_o(0)} = \frac{\bar{I}_b(\tau)}{\bar{I}_c}\ \frac{\tilde{R}(\tau)}{\tilde{R}_o(0)}\ , \tag{22}$$

so that, if α^*, β^*, γ^* are very small

$$Q = \frac{\beta^*}{\gamma^*} \frac{\bar{I}_b(\tau)}{\bar{I}_c} \; Q_k. \tag{23}$$

In the light of the foregoing the method promises adequate results when diligently applied to thin turbulent sheets, as for example to thin free jet boundary layers anchored at the mouth of wind tunnel nozzles. The outside flow is steady, while the turbulent state can be expected to possess the overall persistency required to satisfy condition (1) and thus to permit the linearized approach. Furthermore, close to the nozzle exit the direction $\underline{v}$ can generally be taken as parallel to the known (and constant) direction of the undisturbed flow. The form of the factors of proportionality appearing in relations (20) and (23) depends on the functions chosen to approximate k and k'; however, the mere fact that proportionality exists suffices in all cases where scale effects are irrelevant, e.g., in determining convection speed, which calls for finding that particular spot, $\tau = \tau^*$, at which $R_k(0,0,\tau)$, and therefore $R(\tau)$, become maximal. Strictly speaking one must insure that $\bar{I}_b(\tau)$ does not vary with τ to any significant degree, since by expression (10) and (20)

$$R_k(0,0;\tau) = \frac{1}{4\alpha^*\beta^*} \frac{1}{\bar{I}_a \; \bar{I}_b(\tau)} \; R(\tau).$$

The convection velocity

$$\underline{V}_c = \frac{s}{\tau^*} \underline{v}$$

is usually found somewhat smaller in magnitude than the undisturbed velocity parallel to it.

VOLUME TURBULENCE

The proportionality (14), if proven valid, relates the two-point value for A,B to the two-beam value along a,b. The important role of the direction $\underline{v}$ is to define those two especial points on the a- and b-beams for which the proportionality has a chance to exist. The

strong correlation between the origins will gradually subside as $|\alpha|$ and $|\beta|$ increase. Outside certain "correlation segments" surrounding A and B the contributions of R_k to the value $\tilde{R}(\tau)$ as given by expression (13) will be negligible. These segments can be expected to be relatively narrow if a and b are far from parallel. Still, with unruly turbulence there will be sizable contributions to the integrand (13) by points outside $\alpha = 0$ and $\beta = 0$ with values $R_k(\alpha, \beta; \tau) \neq R_k(0,0;\tau)$ so that the proportionality (14) is precluded.

To gain access to the problem posed therewith let us pass through A a number of a-beams, $a^{(n)}$, in different directions and equally many b-beams, $b^{(n)}$, through B, in directions more or less vertical to the plane containing AB and the corresponding beam $a^{(n)}$. The two-point product mean values $R_k^{(n)}(\alpha, \beta; \tau)$ will usually be different for the several n's except at the points A and B which are common to every pair of beams. Setting

$$R_k^{(n)}(0,0;\tau) = R_k(0,0;\tau),$$

we have the Taylor expansions

$$R_k^{(n)}(\alpha, \beta; \tau) = R_k(0,0;\tau) + \alpha \left.\frac{\partial R_k^{(n)}}{\partial \alpha}\right|_{\substack{\alpha=0 \\ \beta=0}} + \beta \left.\frac{\partial R_k^{(n)}}{\partial \beta}\right|_{\substack{\alpha=0 \\ \beta=0}} + \ldots \tag{24}$$

where the first term is the same for all pairs.

Also, the covariance functions $R^{(n)}(\tau)$ differ in general from each other, as do the functions

$$\tilde{R}^{(n)}(\tau) = \frac{1}{\bar{\bar{I}}_a^{(n)} \, \bar{\bar{I}}_b^{(n)}} R^{(n)}(\tau).$$

(It is again assumed that the means $\ddot{I}_b^{(n)}(\tau)$ do not noticeably vary with τ.)

If, however, the experiments indicate that, for all n, the values of $R^{(n)}(\tau)$ are, although perhaps not equal to, so at least proportional to those of a function $R^{*}(\tau)$:

$$R^{(n)}(\tau) = \kappa^{(n)} R^{*}(\tau) \tag{25}$$

it will also hold that

$$\tilde{R}^{(n)}(\tau) = \lambda^{(n)} R^{*}(\tau). \tag{26}$$

It follows from expressions (13) and (24) that

$$\tilde{R}^{(n)}(\tau) = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \left[R_k(0,0;\tau) + \alpha \frac{\partial R_k^{(n)}}{\partial \alpha} + \beta \frac{\partial R_k^{(n)}}{\partial \beta} + \ldots \right] d\beta d\alpha \; .$$

In order to establish relation (26) the right side must not depend on $\frac{\partial R_k^{(n)}}{\partial \alpha}$, $\frac{\partial R_k^{(n)}}{\partial \beta}$ (nor on the higher derivatives), meaning that α and β must remain very small. In other words, the integral (13) must be zero outside two very small correlation segments $(\alpha_2^{(n)} - \alpha_1^{(n)})$, $(\beta_2^{(n)} - \beta_1^{(n)})$, inside which R_k is practically constant with respect to α and β. The proportionality (26) then assumes the form

$$\tilde{R}^{(n)}(\tau) = (\alpha_2^{(n)} - \alpha_1^{(n)})(\beta_2^{(n)} - \beta_1^{(n)}) R_k(0,0;\tau),$$

so that the relation (25), observed as true experimentally, may now be written as

$$R^{(n)}(\tau) = \mu^{(n)} R_k(0,0,\tau). \tag{27}$$

This is the result desired. Since it holds for all conceivable (a,b)-combinations (for all n), there exist two minute correlation volumes surrounding A and B such that, at any given τ, the R_k-value is practically the same no matter which pair of inside points on two such beams is being correlated. The turbulence is then sometimes said to be isotropic (in the neighborhood of the pair A,B)*.

The beam directions have a bearing on the factor $\mu^{(n)}$ only and can be chosen at will in velocity calculations since, by hypothesis, the maxima of all the functions $R^{(n)}(\tau)$ reside at the same value, $\tau = \tau^*$. Indeed, if one is interested in this value only, the relations (25) may be required to hold merely in a narrow τ^*-environment. The proportionalities (27) would then be true for values close to τ^* only, but that is all that is necessary for the purpose intended.

One may argue that, with a constant bulk velocity direction, it might suffice to establish isotropy normal to it only, and accordingly may restrict the beams $a^{(n)}$ and $b^{(n)}$ to those perpendicular to $\underline{v}$ (and to each other). In the event relations (25) are found to be true, the correlation "discs" surrounding A and B will have small diameters in all directions.

Whether or not the experimentation described in the foregoing can be carried out in practice, will have to be decided by those who would be in charge of it. In any case some experimental means must be contrived to make sure that in a turbulent flow where correlation volumes (or discs) might be sizable, they are in fact not so.

Further experimentation is called for, should the direction $\underline{v}$ not be known beforehand.

For example, one could operate with two parallel beams (not too close to each other) and determine the value R(0), which will be small, unless the plane of the beams is parallel to the bulk velocity when R(0) will have increased to a maximum. The plane's proper attitude can be found by rotating the a-beam about the axis b. A second pair, with a different direction of b, will secure a second plane, also parallel to flow direction; the line of intersection yields the direction $\pm \underline{v}$. Repeated experimentation will show whether or not $\underline{v}$ meets with the requirement of constancy in the turbulent zone investigated.

---

*The usual definition of isotropy refers to one single point and to the time-averaged squares of the velocity fluctuation components in any one direction.

Any plane known to be parallel to $\underline{v}$, but not to the first plane, would alleviate the work load, since it can serve as the second plane. In the study of atmospheric turbulence it is often justified to take such a plane as horizontal.

CONCLUDING REMARKS

Physically, the requirement of small correlation volumes would mean that, if A and B are considered as the centers of eddies, their diameters must be very small. This is hardly surprising; beam correlation can be reduced to point correlation, when the correlating agents are point-like themselves. Moreover, the concept of convection speed has a clear meaning only then. It grows ill-defined with eddies of finite size, since every fluid particle residing in them pursues its own course, sharing a common or bulk velocity component with every other particle. One might be willing to define the latter with the use of that particular value, $\tau = \tau^*$, that maximizes $R(\tau)$, a function no longer proportional to $R_{lk}(0,0;\tau)$, but also no longer required to be so. An eddy particle leaving point A will not as a rule pass through B, nor even cross the line b. The geometric bulk of the eddy, however, will abide in the neighborhood of B after a travel time of about τ^* seconds. In these circumstances experimentation to prove isotropy is not necessary, while the direction vector $\underline{v}$ must be determined and found constant, so that the length AB = s is available for computing the bulk speed.